GAME THEORY AND ITS APPLICATIONS

by

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S Y N O P S I S

This Graduation Exercise is mainly an attempt to study and evaluate the Theory of Games and its applications. Here, greater emphasis is given to the section on the n-person games since this field has not been fully explored by other writers.

In Chapter One, a study is made on the basic concept of Game Theory. The elements involved in the Theory is also discussed in this chapter. Due consideration is also given to the elementary features found in a game.

Chapter Two is devoted to the exposition of the two-person, zero-sum game. This is the simplest game in Game Theory. Here only two parties are considered.

Chapter Three contains the discussion of the applications of two-person, zero-sum games. Business problems are discussed here. An example is also given in this chapter. The latter part of the Chapter deals with non-zero-sum games.

Chapter Four takes up the discussion on the concepts of n-person games. The applications of n-person games is also dealt with here.

In the concluding part of the chapter, four models are formulated and the solutions are found for all the models.

CHAPTER I

GAME THEORY[1] AND ITS APPLICATIONS

What is Game Theory?

Game theory is a method for the study of decision-making in situations of conflict. 2 It deals with problems in which the individual decision-maker is not in complete control of the factors influencing the outcome. A general whose forces face the enemy, an industrialist whose products must complete with those of another industrialist, a finance executive who has to evaluate and select investment proposals, a player in a poker game, duelists, politicians fighting for a nomination and bridge players are all involved in struggles which we may classify as game situations.

The essence of a game problem is that it involves individuals with different goals or objectives whose fates are interlocked. There are many examples of decision-making 3 where this is not so. An architect who has been allotted a specified sum of money in order to carry out a given building program or an engineer engaged in redesigning an industrial process in order to cut cost of production are not involved in a game situation. The engineer and architect face direct minimization or maximization problems in which they are in control of the relevant variables and do not have to contend with anti-engineers or anti-architects who try to destroy their work. The architect may try to maximize certain features of the quality and quantity of building that he can get done for the amount of money at his disposal. The engineer tries to minimize costs for the output of goods required. There may be forces which they do not control, such as the weather; but in most cases some physical influences exist. These are games against nature.

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1. Classical source: Von Neumann and Oskar Morgenstern, Theory of Games and Economic Behaviour, 2nd Edition, Princeton University Press, Princeton, New York, 1947.

2. Mckinsey, J. C., Introduction to the Theory of Games, McGraw-Hill Book Company, New York, 1952, Chapter I.

3. For further details, see Luce, R. D. and H. Raiffa, Games and Decisions, Introduction and Critical Survey, John Wiley & Sons, Inc., New York, 1957.

On the battlefield 4 we may assume that the opposing general is consciously trying to thwart our purposes. The rival firm in business may be actively engaged in taking our customers away from us. In the first case we do not have a game situation; in the other case we do.

The problem of game theory is more difficult than that of simple maximisation. The individual has to work out how to achieve as much as possible, taking into account that there are others whose goals are different and whose actions have an effect on all. A decision-maker in a game faces a cross-purposes maximization problem. He must plan for an optimal return, taking into account the possible actions of his opponents. Where information of competitor's behaviour is available, it is possible to choose that decision which maximizes the firm's expected return after the effects of the rival's counter-moves are taken into consideration. Procedures which resemble this are frequently encountered in business practice. But it is often against the competitor's interests to permit this sort of a calculation. Management may therefore avoid too obvious a pattern in its decision-making in order to keep the opposition guessing. When it succeeds in this goal, no such simple prediction of competitive behaviour will be possible. Thus it is necessary to adopt another approach to the analysis of competitive behaviour. This approach is more deductive. Instead of asking, inductively, what we can infer from the competitor's past behaviour, one seeks to determine a rival's most profitable counter-strategy to one's own "best" moves and to formulate the appropriate defensive measures. This is the approach of game theory.

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4. McDonald John, Strategy in Poker, Business and War: W.W. Norton & Co., New York, 1950.

2. <u>The Elements of Game Theory:</u>

The elements which describe a poker [5] game are the players, money, a pack of cards, a set of rules describing how the games are played, which hands win in any situation that can arise, and what information conditions there are at any stage of the game. The elements which describe the situation of two firms in an advertising campaign are the two sets of individuals in control of the decisions of both firms, the amounts of money available, the information state, the market forecasts of the effect of different types of advertising 6, and the various laws and physical conditions which delineate which actions are legally or physically possible. The situation in which two opposing field commanders may find themselves can be described in terms of the number of men at their disposal, the amount of equipment, their information and intelligence services; the terrain of the battlefield and weather conditions and their valuation of the importance of various objectives.

All the above examples obviously have a common core. A game is described in term of the players or individual decision-makers, the rules of the game, the payoffs or outcomes of the game, the valuations that the players assign to various payoffs, the variables that each player controls, and the information conditions that exist during the game.

These elements, common to all situations of conflict [7], are the building blocks of game theory. They play the same role in this theory as do particles and forces in a theory of mechanics. The players and the rules of the game provide a description of the physical situation and the attempt of the players to maximize or to achieve some individual goal provides the motivation or force.

A player in a game is an autonomous decision-making unit. A player is not necessarily one person; it may be a group of individuals acting in an organisation, a firm, or an army. The feature that distinguishes a player is that it has an objective in the game and operates under its own orders in an attempt to obtain its objective.

Each player is in control of some sets of resources. In poker these resources are cards and money; in business corporations they are various assets; in war, men, armaments, and resources. These resources, together with the rules of the game, describing how they can be utilized, enable us to work out every alternative that is available to a player. In chess we start with a set of pieces placed on the board in a certain manner; the rules tell us how each piece can be moved.

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5. Kuhn, H. W., "A Simplified Two-Person Poker", Annals of Mathematics Studies No. 24 Princeton (Princeton University Press) 1950.

6. For a superb exposition, see "Game Theory Models in the Allocation of Advertising Expenditure", Operations Research, Vol. 6 No. 5, Sept. 1958 (pp. 699-709).

Given that information, it would be possible to work out every possible first move that is feasible in a chess game. As we know the initial distribution of the enemy's men and the rules concerning their movement, we can also work out every possible alternative that he can choose for his first move. In fact, it is theoretically possible to work out the game of chess without every playing it because we could calculate every possible way of playing the game beforehand. Practically the computation problem is too immense to carry out, but we can imagine a game of chess being played in which each player goes up to the referee, hands him a book containing his complete strategy 8 for the game, and the leaves. The referee then works out the game according to these instructions. A strategy for a chess game is a complete set of instructions which states how a player will make every move until the end of the game, taking into account all information concerning the enemy's moves. A strategy in war or in business is the same. It is a general plan of action containing instructions as to what to do in every contingency. Thus, the commanding general may tell his subordinates how he wants the attack to begin, then he may tell them what he wants done after the first part of the attack, depending upon what the enemy's actions have been up to that point.

The outcome of a game will depend upon the strategies employed by every player. Let us call the set of possible strategies that the i th player can use Si. This is the set of every possible plan of action that the i th player can have, taking into account his resources, what he can do with them, and also taking into account every possible act by his opponents. Suppose that the i th player selects a strategy si out of all his available strategies Si. The outcome of the game to him will depend upon what he did and what his opponents did. His payoff is a function of the strategies employed by all the players. We can denote the payoff to the i th player by the payoff function Pi (S_1, S_2, S_3,..., Sn). The possible payoffs in chess are win, lose, or draw; in poker they are various sum of money; in business, profits and growth. In every case each player must have a method of valuation or a utility 9 function which enables him to decide whether or not one payoff is better than another. In business, and in games the payoff may be in money, and there may be no difficulty in distinguishing between a payoff of $1,000 or one of $200. However, in many cases the payoff can be complicated by other factors.

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7. Williams, J. D., The Compleat Strategyst, McGraw-Hill Book Company, New York, 1954.

8. William, J. D., ibid.

9. Swalm, "Utility Theory", Harvard Business Review, 44, pp. 124-25

For instance, the payoff arising from the following one line of action in battle may result in 1,000 enemy casualties at a cost of 200 men lost; another line of action, could result in 5,000 enemy casualties at a cost of 2,000 men lost. It is difficult to say which is preferable.

In general, a player has a valuation scheme 10 by which he can evaluate the worth of any set of prospects with which he is confronted. For instance, we assume that a player knows whether or not he would rather make a profit of $10 million or $1 million. The game in which he is playing may be such that he can never obtain a profit of $10 million. This amounts to saying that the prospect of a profit of $10 million to a player is not a possible payoff in this game.

We may now reformulate the problem of game theory. An N-person[11] game consists of a set of N players, each in control of a set of strategies Si, i = 1, 2, ..., N; each player has a payoff function Pi (S_1, S_2, ..., SN) which tells him what prospect he receives as his payoff if each player has chosen his strategy si. The object of every player is to attempt to obtain a payoff which yields him a prospect of maximal value.

The technical terms described earlier give us a method whereby we can formalize any sort of situation involving conflict. For the general purposes of the game theorist this is very desirable. Those of us interested in management must ask: Can the general scheme be applied to areas of specific interest to us? It turns out that the simplest sort of game we can discuss has several useful applications.

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10. Martin Shubik, "Game Theory and Related Approaches to Social Behaviour."

11. Shapley, L. S. "A Value for N-People Games", Contribution to the Theory of Games, Vol. II Princeton: Princeton University Press, 1953, pp. 307-318.

CHAPTER II

THE TWO-PERSON, ZERO-SUM GAME

Basic Conception

The Two-Person Zero-Sum Game:

This is a game in which the amount that one player loses is precisely the amount that the other player wins. Two-person poker, matching pennies, and most other two-person games are of this variety. Consider a competitive struggle for market share by two firms (duopoly) assuming there is no price war and collusion. Here, every percentage point gained by one of the firms is necessarily lost by the other. Here, we have a two-person, zero-sum game. It is called zero-sum because, no matter what is done by either competitor, the total gain in market share to the two players is zero, so that the interests of the competitors are diametrically opposed. In this example, so long as their number remains unchanged, the share of the market which the two firms have between them is necessarily 100 per cent [12]. But the game is zero-sum only because of the nature of the company objectives. Decisions which are taken by the firms may very well increase or reduce both the absolute size of their market and the total profits of the two taken together. Increased advertising[13] outlay and price-cutting may, for example, increase their combined sales volume and reduce their profits, and if the firms' objectives were maximization of either profit or sales rather than market share, the game would not be of the zero-sum variety. However, there are situations in business and war which can be approximated by a zero-sum model.

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12. Strictly speaking, this is a "constant sum" rather than a zero-sum game because the sum of the two market shares is the fixed number 100 (per cent), not zero. However, there is no significant analytic difference between the constant sum and the zero-sum games. This is because there is no change in strategic possibilities from a given game to another game in which some constant amount that cannot be changed by the players is added to the original payoffs.

13. For further details, see Charnes, A., and Cooper, W. W., "An Example of Constrained Games in Industrial Economics," (Abstract), Econometrica, 22. Oct. 1954, p. 526.

The relevant features of a two-person zero-sum game can be displayed by making use of a payoff matrix. To illustrate the central concepts of the two-person zero-sum game, let us assume an example. Assume a smuggler has two possible routes over the border: one is down the highway and the other through the mountains. If he could go down the highway unhampered, he could take a fully loaded truck and make a tidy profit. If there is a light police guard on the main road, he can avoid arrest but will not be able to get his load through and will have to lose the expense of the journey. If there is a heavy police guard on the road, he will be caught and will be arrested and lose his load. The mountain road is such that he can only take a small load. If it is unguarded, he will have no trouble. If it is lightly or heavily guarded, then he can still get through but will have to bribe the peasants to get him by the police. The police have three alternatives: they can put a heavy guard on the main highway, leaving the mountain route unpatrolled; a heavy watch on the mountain route, leaving the highway unpatrolled; or to split their forces and put a light guard on both.

We can display the smuggler's values for the six possibilities as follows:

	Guards only Highway	Guard both routes lightly	Guard only Mountain road
Highway	-5	-2	5
Mountain Road	2	1	1

The negative numbers represent losses and the positive numbers represent gain.

The police's preferences are diametrically opposed to those of the smuggler; thus, their valuation for any outcome is the negative of his. We call this type of game strictly determined because, upon examination of the payoffs, there is a definite optimal choice for the smuggler which is to take always the mountain road, while there is also an optimal choice for the police which is to guard both roads lightly. Both sides can work out that they can always enforce this compromise on the other but can enforce nothing better. The smuggler knows that if he chose the highway, the police would try to minimize his gain and could guard it heavily; if he chose the mountains, the police would try to minimize his gain and guard the mountains; but even if they did so, the worst that could happen to him is that he would be able to get a small shipment in after having bribed the peasants.

The police argue that if they decided to guard the highway only, the smuggler would use the mountains; if they guarded the mountains only, he would use the highway; if they guarded both lightly, he would use the mountains but would only be able to get a small shipment by them, no matter what he did. We can illustrate these computations by adding a column giving the minimum of each row in the matrix, and a row giving the maximum of each column:

	Strategy of Police			
Strategy of smuggler	1	2	3	row Minima
1	− 5	− 2	5	− 5
2	2	1	1	1
column maxima	2	1	5	

The column represents the computation done by the smuggler which tells him what the police sould do to him if he chose strategy 1 ar 2. The row presents the computation done by the police on the assumption that the smuggler would try to maximize against their actions. By choosing the mountains, the smuggler guarantees for himself the maximum of the ~~minimum~~ minima. By putting a light guard on both roads the police guarantees that the smuggler can never get more than the minimum of his maxima. But we observe that here the

$$\text{Minimax} = \text{Maximin} = 1$$ [14]

The smuggler can guarantee a small trade for himself, and the police can guarantee that it stays a small trade, no matter what the other side does. A game which has the property that each side has a strategy which results in the maximin, being equal to the minimax is said to possess a saddle-point. [15]

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14. Based on the Minimax Theorem.

15. Von Neumann first proved this.

An economic interpretation can be given to this value. When the smuggler decides to retire, the market value of his trade should be that amount which yields an income of 1 in the same period as it takes per trip.

Not all games possess saddle points, and in those which do not, it is not possible for one side to pick a strategy which guarantees very much. For instance, imagine that there had been a general overhauling of both smuggling and police techniques. The smuggler had obtained better trucks, and the police managed to stop bribery in the mountains. The effect of the better trucks is that the smuggler can get by a light police guard at the cost of some breakages and personal strain. He now can carry a bigger load both on the highwa y in the mountains. The effect of the police improvement is that a strong patrol could catch the smuggler if he were in the mountains. The new payoff matrix is:

	Strategy of Police			
Strategy of smuggler	1	2	3	row minima
1	-5	3	6	-5
2	6	3	-5	-5
column maxima	6	3	6	

The new matrix values are assumed as above. This is the class called degenerate solutions. Here we have mixed strategies. If the smuggler persists in sticking to the mountain route, he will be lost; if he keeps to the highway, he will be lost. There is no longer a simple decision to which he can commit himself which will yield him a guaranteed profit, even though his techniques seem to have improved more than those of the police. He still has a profitable trade, however, and there is a way for him to guarantee himself an expected profit by following the actions and precepts of most decision-makers in competitive trades, and that is to take a calculated risk 16. His problem is to decide how to calculate the risk he should take. He is a prudent man* and has no false illusions about the stupidity of the police. He knows, for instance, that the police will never split their forces because this would amount to handing him an income of three per period, no matter what happened.

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16. Hertz, B. D., "Risk Analysis in Capital Investment", Harvard Business Review. Feb. 1964.

The smuggler wishes to calculate what is the biggest expected income that he can guarantee for himself, regardless of what the police tries to do. At the very worst, they could find out his plans and maximize their return, i.e. minimize his return given this information. If he definitely commits himself to one action and plays a pure strategy, he stands to lose 5. He may, however, decide not to commit himself directly but to choose between his two pure strategies according to some probability 17 weighting. We call the use of such a device, which attaches probability weightings to a set of pure strategies, a mixed strategy . By using a mixed strategy he can guarantee an expected profit, even if the police were to find out his strategy.

Suppose he decides to take the highway with a probability of x_1 and the mountains with a probability of x_2, where $x_1 + x_2 = 1$. He wishes to pick these numbers in such a manner that he can make his expected return, which we call V, as large as possible under all circumstances. 18 If the police employed their strategy one against him, his expected return would be: $-5x_1 + 6x_2$. This must be greater than, or equal to, V. Similarly, for the other strategies, we can write down an inequality. We find that we must solve the following set of equations and inequalities:

$$
\begin{aligned}
-5x_1 + 6x_2 &\geq V \\
3x_1 + 3x_2 &\geq V \\
6x_1 - 5x_2 &\geq V \\
\text{and} \quad x_1 + x_2 &= 1
\end{aligned}
$$

Similarly, the police wish to make sure that no matter how shrewd the smuggler becomes they will be able to restrict his expected gains as much as possible. In fact, they can make sure that he never can get more than an expection of V. The police decide to guard the highway with probability y_1, split forces with probability y_2, and guard the mountains with probability y_3, where $y_1 + y_2 + y_3 = 1$. The police must solve the following set of equations and inequalities:

$$
\begin{aligned}
-5y_1 + 3y_2 + 6y_3 &\leq V \\
6y_1 + 3y_2 - 5y_3 &\leq V \\
\text{and} \quad y_1 + y_2 + y_3 &= 1
\end{aligned}
$$

In linear algebra this is the dual of the primal stated earlier. So also in linear programming.

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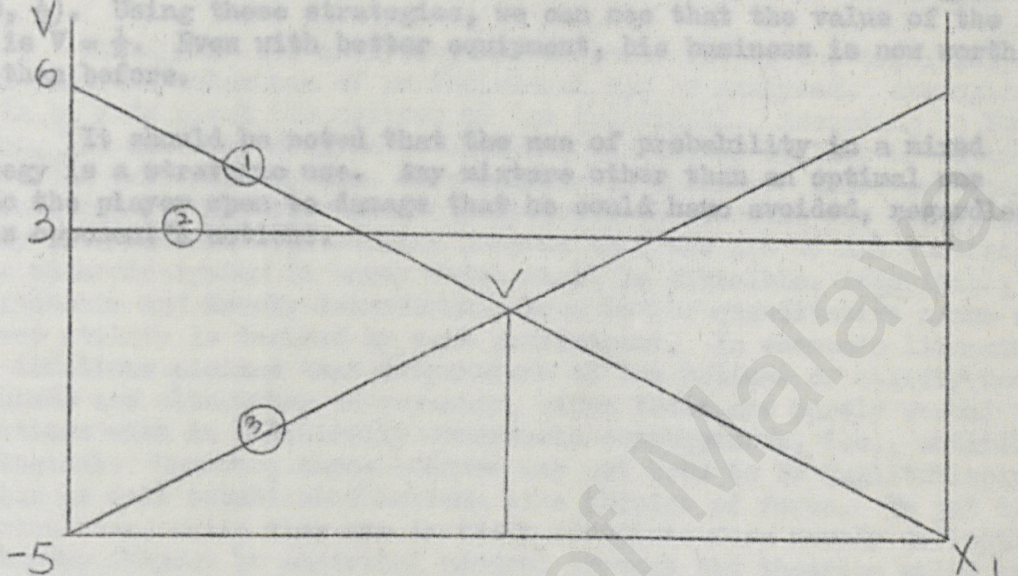
17. Hein, L. W., "Quantitative Approach to Managerial Decisions", pp. 144-134.
18. This is the point where the Neumann-Morgenstern utility theory is required. For a discussion of the theory, see Appendix to this chapter.

The general method for the solution of such systems can be found in Mckinsey's book 19, a graphical method for this 2 x 3 game may be presented.



Graph I

If the first player uses the probabilities of x_1 and $1 - x_1$ and if the second player uses his first pure strategy, then the expected payoff for the first player is

$$-5x_1 + 6(1 - x_1) = 6 - 11x_1;$$

Similarly, we got two other expressions if the second player uses his second or third pure strategies. We now draw a diagram with the 3 lines: $v = 6 - 11x_1$; $v = 3$; and $v = 11x_1 - 6$ represented over the interval $(0, 1)$. For any x_1 chosen by the smuggler he can guarantee for himself the minimum value of the three lines at x_1; thus we see here that the optimal choice for x_1 is $\frac{1}{2}$. Hence, his mixed strategy uses each of his pure strategies with probability of $\frac{1}{2}$ and can be expressed as $(\frac{1}{2}, \frac{1}{2})$

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19. Mckinsey, J. C., Introduction to the Theory of Games, Mcgraw-Hill Book Company, New York, 1952, Chapters 2 and 3.

It is clear that the police will never use their second strategy; thus, we need only investigate the probability weighting of the police to guard the highway or the mountain road. Their optimal strategy is $(\frac{1}{2}, 0, \frac{1}{2})$. Using these strategies, we can see that the value of the game is $V = \frac{1}{2}$. Even with better equipment, his business is now worth less than before.

It should be noted that the use of probability in a mixed strategy is a strategic use. Any mixture other than an optimal one leaves the player open to damage that he could hawe avoided, regardless of his opponent's actions.

APPENDIX TO CHAPTER II

1. THE IDEA OF RATIONAL BEHAVIOUR

Throughout our analysis, we have made use of the traditional nòtion that the behaviour of an individual can be analysed. Assumptions have to be made about the motives of the individual. According to Von Neumann, the consumer desires to obtain a maximum of utility or satisfaction and the entrepreneur aims at a maximum of profits. There are conceptual and practical difficulties when one attempts to describe the notion of utility as a number. Von Neumann assumed that the aim of all participants in the economic system is money which shall be divisible, indentical, substitutable and freely transferable even in the quantitative sense with whatever utility is desired by each participant. In economic literature, it is sometimes claimed that discussions of the notions of utility and preference are altogether unnecessary, since these are purely verbal definitions with no empirically observable consequences, i.e., entirely tautological. However, these notions may not seem to be qualitatively inferior to well established notions like physics of force. To put it in another way, while they are in their immediate form merely definitions, they become subject to empirical control through the theories which are built upon them. Hence the idea of utility or satisfaction is raised above the status of a tautology by such economic theories as make use of it.

The individual, according to Von Neumann, who attempts to obtain these respective maxima is also said to act "rationally". At present there is no satisfactory treatment of the question of rational behaviour. For example, there are several ways to reach the optimum position depending on the knowledge and understanding which the individual has and upon the paths of action open to him. Qualitative study of all these questions may also imply quantitative relationships. In order to take into consideration all the elements of the qualitative description, it would be necessary to formulate them in quantitative terms. This is not an easy task, and so far, it has not been successful. The main reason is that suitable mathematical methods have not been developed and applied to the problem. This goes to show that the maximum problem need to be formulated in a mathematical way. The more recent approach in dealing with individual's choices is in the form of indifference curve analysis or ordinal analysis. This is held to be more superior in many ways.

2. <u>The Utility Theory</u>

According to the "Lausanne" theory, attention must be paid to the interdependence of the participants in a social economy. In utility, the immediate sensation of preference — of one object as against another — provides the basis for utility to be conceived as a quantitative measurement. But this permits us only to say when for one person one utility is greater than another. It is not in itself a basis for numerical comparison of utilities for one person nor of any comparison between different persons. Since there is no intuitively significant way to add two utilities for the same person, the assumption that utilities are of non-numerical character even seems plausible. The modern method of indifference curve analysis (ordinal analysis) is a mathematical procedure to describe this situation.

Let us consider, then, the fact that a person will prefer certain events to others, and that in other cases, he will be indifferent as to two events. This gives us the two relations, "p" and "i". The domain of these relations will be the set of events. Let us use A, B, C, etc. to denote events.

(i) <u>Definition</u>

Given any two events, A and B, we say A p B if A is preferable to B. We say A i B if A $\not p$ B and B $\not p$ A.

(ii) <u>Utility Axioms</u>

The relations "p" and "i" satisfy the following axioms:

(i) Given any two events, A and B, exactly one of the following must hold:

$$\left\{\begin{matrix}(\alpha) \\ (\beta) \\ (\gamma)\end{matrix}\right\} \quad \begin{matrix}A\ p\ B \\ B\ p\ A \\ A\ i\ B\end{matrix}$$

(ii) A i A for all A.

(iii) If A i B, then B i A.

(iv) If A i B, and B i C, then A i C.

(v) If A p B and B p C, then A p C.

(vi) If A p B and B i C, then A p C.

(vii) If A i B and B p C, then A p C.

Axiom (i) is also called the trichotomy law.
Axioms (ii), (iii) and (iv) mean that i is an an equivalence relations. Axiom (v) (together with Axiom (i)) means that p is an order relation. Finally, Axioms (vi) and (vii) are sometimes described by saying that p is transitive over i. Essentially, the Axioms here represent a weak linear ordering of events, from the most desirable to the least desirable. It is usual to say that, if someone prefers the event A to the event B, then the event A has more utility than B, or the utility of A is greater than that of B.

Unfortunately, while this tells us that the utility of A is greater than that of B, it does not tell us how great this difference in utility is. For certain problems, e.g. if it is merely a question of determining which event an individual will choose, this presents no difficulty. On the other hand, if any risk is involved, it is clear that we must have some idea of the difference in utility between a given pair of events (and not merely which one is preferred by an individual). If, for instance, a person must choose between the event B, and a lottery which will give either the event A or the event C with equal probabilities, and if, moreover, A p B p C, then the problem is to determine whether the possibility of gain (in case A happens) is enough to offset the risk of loss (in case C should occur).

If, now, there existed some commodity (call, perhaps, utiles) for which utility were linear (i.e. such that the utility of a certain quantity of utiles were directly proportional to that quantity) there would be no difficulty. We could simply decide the size of the side-payments (in utiles) which would induce our player to abandone A for B, and B for C, respectively, and then act accordingly. Unfortunately, we have seen that no commodity (not even money) behaves in this manner, and thus this idea is deficient. Yet we may show that such a commodity can be dealt with and considered, just so long as we do not attempt to deal with it as with other commodities which can be bought, sold, transferred, or destroyed.

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3. Case of the Lottery

Risk-taking is one of the principal cases in which utiles may be dealt with. Now, risk-taking depends essentially on the idea of a lottery.

Definition: Let A and B be any two events, and let $0 \leq r \leq 1$ Then by $rA + (1 - r)\, B$ we mean the lottery which has the two possible outcomes, A and B, with probabilities r and $1 - r$, respectively.

In a similar way, lotteries with three or more possible outcomes maybe defined. It should be pointed out that a lottery is also an event. The combination of events by means of lotteries obeys all the usual laws of arithmetic and linear algebra. We have the following Axioms:

(1) $rA + (1 - r)B = (1 - r)B + rA$

(2) $rA + (1 - r)\left\{sB + (1 - s)\,C\right\}$
$$= rA + (1 - r)\,sB + (1 - r)\,(1 - s)C$$

(3) $rA + (1 - r)\,A = A$

Axiom (1), a commutative law, has an obvious meaning. Axioms (2) and (3), which resemble the distributive law, state that the manner in which a lottery is held does not matter: only the final probabilities of the possible outcomes matter.

Now, in the lottery $rA + (1 - r)B$, it is clear that it should not make any difference if A is replaced by some C such A i C. We have thus the Axioms.

(4) If A i C, then, for any r, B,
$$\left\{rA + (1 - r)B\right\} \;\; i \;\; \left\{rC + (1 - r)B\right\}$$

(5) If A p C, then, for any r 0, B,
$$\left\{rA + (1 - r)B\right\} \;\; p \;\; \left\{rC + (1 - r)B\right\}$$

When we consider the lottery, $rA + (1 - r)B$, we see that, for $r = 1$, the lottery is identical to the event A; while for $r = 0$, the lottery is identical to B. Now it seems reasonable that a very small change in r should cause only a very small change in the utility of the lottery.

This, together with the intermediate value theorem for continuous real-valued functions, gives the Axiom:

Continuity Axiom. Let A, B, and C be events such that A p C p B. Then there exists some $r \in [0, 1]$ such that

$$\{rA + (1 - r)B\} \text{ iC}$$

It should be remarked that, in Axiom above, we must actually have $r \in (0, 1)$. In fact, if r = 0 or 1, the lottery is equal to B or A, respectively, and by hypthesis A p C p B, so that we cannot have a lottery equivalent (under i) to C in either of these two cases. This can be stated as a theorem.

If A p C p B, and $\{rA + (1 - r)B\}$ iC then $0 < r < 1$
moreover r is unique

The axioms which we have given are now sufficient for the construction of a utility function.

CHAPTER III

APPLICATIONS OF TWO-PERSON ZERO-SUM GAMES

1. Business Problems

The type of problem to which this theory has a direct application is one which has some of the aspects of a duel. A duel has the property that the goals of the opponents are diametrically opposed. In any market in which the size of the demand is more or less fixed by the government or by habits, the extra customers that one firm can attract must have been lost by another firm. The firms are in pure opposition in such a situation. What one gains, the other loses. An advertising campaign[20] in a market for petroleum products maybe of this nature. A highly simplified example of our local environment may be cited. Charles and Cooper have written a more detailed paper on this topic.[21]

Two oil companies, Shell and Mobil, each have a million dollars to spend on advertising their products in a certain market area. They can use the media of radio, television, newspapers, magazines and billboards. For simplicity, they can be grouped into radio, television, and printed media. The marketing research sections of each firm work out the expected effect of any contingency. We will discuss the decision-making at Shell company only. A payoff matrix of 4 x 4 is drawn up. This contains information on the 16 contingencies that might arise if ~~their~~ either company spent all its money advertising solely by means of radio, television, or printed media, or decided to save the million dollars and not advertise at all. Each entry in the payoff matrix represents the amount of extra revenue above cost estimated under these circumstances (in million of dollars) for Shell.

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20. For details and models,

    see "Game Theory Models in the Allocation of Advertising Expenditure", Operations Research Vol. 6, No. 5, 1958, (pp. 699-709).

21. Charnes and Copper., ibid.

		MOBIL			
		Radio	T.V.	Printed Media	No Advertising
SHELL	Radio	0	-0.5	0	2.5
	T.V.	2	0	1.5	5
	Printed Media	1	-0.5	0	3.5
	No Advertising	-2	-4	-3	0

We can see immediately that in this case the alternative of no advertising can be rejected. Any pure strategy, which can be rejected by comparing it with the other pure strategies and finding that there are others which are always better under every circumstances, is a dominated strategy and will not enter into a solution. In this example, where we have assumed that the firms must put all their money into one advertising medium, we can see by inspection that all the other strategies are dominated by television. This game has a saddlepoint at which both firms put their money into television advertising with the net result that they make the same as they would if neither advertised, but neither can risk not advertising. This refers to Collusion.

A more complicated and slightly more realistic example is obtained if we list a series of advertising campaigns involving different integrated programs using more than one medium. Consider each company to have the choice of three types of campaigns.

		MOBIL		
		Program 1	Program 2	Program 3
SHELL	Program 1	2	4	-2
	Program 2	4	2	-2
	Program 3	-2	-2	3

The problem for Shell Company is to find three numbers, x_1, x_2, x_3: such that

$$2x_1 + 4x_2 - 2x_3 \geq V$$
$$4x_1 + 2x_2 - 2x_3 \geq V$$
$$-2x_1 - 2x_2 + 3x_3 \geq V$$

where the

$$x_i \geq 0 \quad \text{and } x_1 + x_2 + x_3 = 1$$

This example has a solution of $x_1 = \frac{1}{4}$, $x_2 = \frac{1}{4}$, $x_3 = \frac{1}{2}$, and $V = \frac{1}{2}$.

Two interpretations can be given to the x_3. They can be regarded as probabilities which should be attached to the decision to adopt any specific program. Or, if it is possible to spend varying sums on the programs (with roughly constant returns), then x_i give information as to how Shell Company should split up its advertising budget between the three different programs. It should spend \$250,000 on each of programs 1 and 2 and £\$500,000 on program 3.

An incomplete expenditure in each program might result in poorer outcomes in matrix and the whole configuration can change. The more satisfactory way to treat the advertising problem is as one of a series of games being played every period. Charnes and Cooper suggest this approach in their analysis of "Constrained Games" in their article noted earlier. Leonard Gillman [22] of Massachusetts Institute of Technology has also written an article on advertising competition. In this article, he introduces the concepts of susceptibility and resistance functions as used in his mathematical model to determine the strategy and spending rates of the participants in the advertising campaign.

Competition between two refineries sharing a market with relative fixed demand has been set up and treated as a two-person zero-sum game by G. H. Symonds[23] in his examination of game theory uses in problems of petroleum refining.

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22. Leonard Gillman, Operations Analysis and the Theory of Games, American Statistical Association Journal, 1950 pp. 541-545.

23. Symonds, G. H., "Applications to Industrial Problems", (abstract), Econometrica, 22, 1954, p. 526.

Considerable work has been done in the application of the game theory of "duels" to problem of weapons evaluation for tactical weapons. These applications are of more interest to those whose scope is confined to operations research. Much of the work in this area is classified. Search theory has, however, been applied to marketing.

Gale, Kuhn, and Tucker[24] have discussed the mathematical analogy that exists between the solution of a linear program and the solution of a two-person zero-sum game. It is always possible to formulate such a game from a linear program in such a manner that the solution of this game amounts to a solution of the linear program.

Assuming we are faced with a problem of filling n jobs with n people. Further assume we have an evaluation a_{ij} which tells the ~~work~~ worth of the i th person doing the j th job. The optimal assignment problem[25] concerns itself with the distribution of personnel in a maximal manner. There is such a related game where solution gives us a solution of this problem. The difficulty in this application comes in the evaluation of the suitability of the attributes of various individuals in the performance of difference tasks. This method has been used to a limited extend by the Army.

There have been situations in which sampling or gather extra information costs money, yet cuts down on the possibility of making a wrong decision which may, in itself, be very costly, lead to the formulation of statistical games. In essence, the problem amounts to working out how much one should be willing to pay for information, the value of which will not be known until it is obtained.

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24. Gale, D., Kuhn, and Tucker, A. W., "Linear Programming and the Theory of Games", in Koopmans, ed., Activity Analysis of Production and Allocation, John Wiley & Sons, Inc. N. Y. 1951.

25. Von Neumann, "A Certain Zero-sum Two-Person Game Equivalent to the Optimal Assighment Problem," in Kuhn and Tucker, ed., Contributions to the Theory of Games, Vol. II, Princeton University Press, Princeton, 1953, pp. 5 - 12.

To take an important example from industry, is the design of a decision process to be followed in sequential sampling of a batch of goods where the cost of sampling is high and the loss incurred by sending out a batch with above a certain number of defective items is great.[26]

2. Non-Zero-Sum Games:

Many of the more interesting problems of competition are not zero-sum. The goods of a group of large firms in a market are not necessarily diametrically opposed. There may be room for all if instead of fighting among themselves they follow a policy of live and let live. A period of cut-throat competition might hurt all of them. When pure opposition of interests is no longer the case; the computations of the two-person zero-sum game theory no longer apply. Unfortunately, the theory is in a far less satisfactory state outside the area of the two-person zero-sum game. In the literature, non-zero sum games are divided into co-operative and non-co-operative games depending on whether collusion does or does not occur.

In the co-operative case[27] the game theorists have tended to argue that the players will be sufficiently rational to discover and make full use of all opportunities which can be mutually advantages. That is, the players are taken to co-operate on any and every action which can increase the payoff of either player (provided it does not, at the same time, reduce the payoff of the other). In the terminology of the neoclassical economist, they will always end up somewhere on the contract curve.

It is however doubtful whether players are really so rational in practice. Further, there are problems involved in arriving at an acceptable division of the payoff which might prevent players from maximising their total loot as this rationality assumption requires. It may be noted that most of the novelty in the co-operative case analysis occurs in investigation of the division of the spoils between colluding players.

Non-co-operative, non-zero-sum games possess a number of interesting features:

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26. Blackwell, D., and Girschick, M., Theory of Games and Statistical Decisions, New York: John Wiley & Sons, Inc., 1954.

27. See R. Aumann, "A Survey of Co-operative Games Without Side Payments", Mathematical Economics of Oskar Morgenstern, pp. 3-24.

(1) The existence of several equilibrium pairs of strategies in a game does not mean that they will all yield the same payoff. For example, if (a, b) and (a^1, b^1) are equilibrium pairs, neither (a, b^1) nor (a^1, b) need be equilibrium pairs. This may contribute to the complication of the planning problems of both players since both may lose out if they do not aim for the same equilibrium pair.

(2) In the zero-sum case, secrecy in planning is an advantage; in the non-zero sum case, it will often pay a player to reveal his plans. This disclosure maybe used as a threat or a means for transmitting information which permits a degree of tacit collusion;

(a) Threat information: To a player who announcesthat he will drop a bomb which will blow everyone up if he does not have his way, disclosure of this information is necessary for him to win his point. Many economic examples such as strike threats, may be cited. [28]

(b) Information for quasi-collusion: For example, a company may publicize any price increases in the hope that this move will soon be followed by other firms in the industry, to their mutual advantage.

(3) Another feature of the non-cooperative non-zero-sum case is that both players will often be led by self-interest to take decisions which are mutually disadvantages. This has been illustrated by a game called the prisoners' dilemma which is attributed to A. W. Tucker. Two prisoners are brought in and interrogated separately. Each knows they will both get off if neither prisoner "talks". However they are both told that if one confesses and the other does not, the one who fails to confess will receive a particularly heavy penalty. In this case both players may well decide to protect themselves by confessing.

This point is of considerable economic importance. It reveals why citizens may not contribute taxes ~~by~~ voluntarily even though each wants the government to function - the citizen sees nothing to be gained by paying taxes unless there is some guarantee that others will contribute too, just as one prisoner will confess unless he has some assurance that his fellow prisoner will do so. Similarly, many storekeepers will keep their shops open on Sunday although they all prefer a holiday, each fearing that if he does not do so he will lose customers to his competitors. This argument is involved in the logic behind conscription and rationing in wartime, government measures to combat inflation, etc.

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28. For a highly suggestive analysis of this and other related problems, see T. C. Schelling, The Strategy of Conflict, Harvard University Press, Cambridge, Mass., 1960.

CHAPTER IV

1. Some Concepts of N-Person Games

Since most industries contain more than two firms, the theory of n-person games is of most widespread potential economic application. This is also true when we consider real international trade problems where many countries are involved. But the theory of n-person games have so far proved rather intractable to analysis. Writings on the subject and results have been fewer than in the case of the two person, zero-sum game which we have discussed earlier. There is nothing in n-person theory resembling the well-rounded analysis of the two-person theory.

Nevertheless, the literature is rich in suggestive ideas - definitions and concepts rather than theorems. Some, but not all of these concepts are matters of common sense and common observation and it is only remarkable that they were given little attention in pre-game-theoretic economic theory. So far, in economic application, such suggestive concepts have been the most fruitful aspect of game theory - they have served to provide a basis of looking at difficult problems rather than a source of ready calculations. Thus, it maybe stated that the present state of n-person theory is rather limited.

In considering n-person games (where $n \geq 3$), the same distinction will arise in considering two-person, non-zero-sum games. That is, games are again divided into the co-operative and non-cooperative varieties. In the non-cooperative case, the principal question is the existence of equilibrium n-tuples. The following theorem can provide a solution for the question.

Theorem: Any finite n-person non-cooperative game has at least one equilibrium n-tuples of mixed strategies.

The proof of this theorem will not be given. It maybe pointed out that all the difficulties found in equilibrium points of bimatrix games [29] are also present here. Furthermore, the computation of equilibrium n-tuples (where $n \geq 3$) is comparably more difficult than the computation of equilibrium pairs. In general, there is no great difference between the theory of non-cooperative n-person games and non-cooperative two-person general sum games.

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29. A finite two-person non-zero-sum game can be expressed as a pair of m x n matrices, $A = (a_{ij})$ and $B = (b_{ij})$, or equivalently, as an m x n matrix (A, B) each of whose entries is an ordered pair (a_{ij}, b_{ij}) The entries a_{ij} and b_{ij} are the payoffs (in utilities) to the players I and II respectively, assuming they choose their ith and j th pure strategies respectively. This is a bimatrix game.

Nash [30] has proved that every non-cooperative game in which each player has only a limited number of strategic alternative open to him has at least one (mixed or pure strategy) equilibrium point. To put it in another way, there exists at least one combination of mixed or pure strategies (a, b, c, ... n) such that if they are employed by players (A, B, C,. . N) respectively, it will be unprofitable for any one of these players to switch to any other strategy. Thus, if all players but one follow this pattern the self-interest of the remaining player will also lead him to stick to the equilibrium pattern.

The difficulties which arise in n-person game are similar to the two-person, non-zero-sum case. Different equilibrium points may yield different payoffs to the players, and if some players aim for one equilibrium point and the remaining players aim for another, they may all end up at a non-equilibrium point. Thus, an n-person game may possess more than one equilibrium point. In the absence of co-ordination of the plans, if there are a number of equilibrium points, the players may find it difficult to attain any one of them - cycling.

1. In co-operative games, however, we find that a new idea appears: that of coalition. In the n-person case, there are possible coalitions and this means that, if a coalition is to form and remain for some time, the different members of the coalition must reach some sort of equilibrium or stability. Thus, for an n-person game, we shall let N = (1, 2, ..., n) be the set of all players. Any non-empty subset of N (including N itself and all the one-element subsets) is called a coalition. A game of the relatively uninteresting variety in which there is no motivation for coalition formation is called an inessential game, as contrasted with essential games in which its members can benefit from the formation of a coalition. Thus, a game v is essential if

$$v(N) > \sum_{i \in N} v(\{i\})$$

It is inessential otherwise. It is the essential games which are of interest to us.

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30. Nash, "Non-Cooperative Games", Annals of Mathematics 54(1951), pp. 286-295.

2. In order for a coalition to maximize its returns, it may be necessary for a member to undergo some sacrifice, For example, a cartel may find it profitable to close the inefficient plant of one of its members rather than getting every member to reduce his scale of operations. In this case, in order to induce the short-changed individual to serve the interests of the coalition it is necessary to set up an equalization payment (bribe) for him. In game theory such a redivision of the spoilts is called a side-payment.

3. An imputation for the n-person game V is a vector $x = (x_1, \dots x_n)$ satisfying:

(i) $\sum_{i \in N} x_i = v\,(N)$

(ii) $x_i \geq v(\{ i \})$ for all $i \in N$

4. But how does one determine how much will be paid to each player in different circumstances? Given the coalition which are formed, what payoffs will be received by the members of each coalition? Von Neumann and Morgenstern approached this question by describing lower limits to those amounts. By the characteristic function of an n -person game we mean a real-valued function, v, defined on the subsets of N, which assigns to each $S \subset N$ the maximum value (to S) of the two-person game played between S and N-S, assuming that these two coalitions form. Thus, v (S) is the amount of utility that the members of S can obtain from the game, whatever the remaining players may do. Thus, it follows that

$$v\,(\emptyset) = 0$$

Now if S and T are joint coalitions, we have the super-additivity property:

$$v\,(S \cup T) \geq v\,(S) + v\,(T) \text{ if } S \cap T = \emptyset$$

Thus, by an n-person game in characteristic function form is meant a real-valued function v, defined on the subsets of N, satisfying the above two conditions.

5. Let x and y be two imputations and let S be a coalition. We say x dominates y through S (notation, $x \leftarrow_S y$)

(i) $x_i > y_i$ for all $i \in S$

(ii) $\sum_{i \in S} x_i \leq v(S)$

We say x dominates y (notation, $x \succ y$) if there is any coalition S such that $x \succ_S y$. Thus condition (i) states that the members of S all prefer x to y; condition (ii) states that they are capable of obtaining what x gives them.

6. The set of all undominated imputations for a game, v, is called the core. The notation for the core is c (v).

Thus the core of a game, v, is the set of all n-vectors, x, satisfying

(a) $\sum_{i \in S} x_i \geq v(S)$ for all $S \subset N$

(b) $\sum_{i \in S} x_i = v(N)$

7. Solution. Von Neumann defined a solution of an n-person game as a set of imputation which has the following characteristics.

(i) If F and G are any two of the imputations in a solution, then neither F dominates G nor G dominates F.

(ii) If I is an imputation which is not included in the solution set, then there is at least one imputation, I*, which dominates I and which is included in the solution. Thus, a solution consists of a set of imputations none of which dominates any other, and which can among them dominate any excluded imputation.

19 (12, 30)

2. Application of n-Person Games:

(i) Construction of Models

For a marketing situation, mathematical models are developed expressing the relation between net revenue or net returns and the variables that affect it. Here we will deal with multicompetitors (i.e. n-competitors) and discuss four models when the market potential is independent of both price and promotional effort and when it is dependent on either or both of the controllable variables.

In a competitive world, sales of goods and services depend upon many factors, like quality, packaging, price, promotion, distribution channels, competition, and marketing strategies.[31] Of these variables some are under one's control, some are under the control of competitors, and others are exogenous variables such as population and national income. Let us assume that to increase sales volume, one can either decrease price or increase promotional efforts. Whether such a strategy is worth pursuing or not depends not only on the competitors' strategies, but also on the characteristics of the market.

Each competitor manipulates his controllable variables so as to maximize his objective function. Here, under the assumption stated below, we can develope four mathematical models expressing the relation between net revenue and the variables that affect it. The assumptions underlying these models are:

1. There are n (6) competitors $x_1, x_2, \ldots, x_n$ and each competitor has two controllable variables, viz., price and promotional effort.
2. Manufacturing cost of each competitor is directly proportional to the production quantity and each competitor knows manufacturing costs of all other competitors.
3. The effectiveness of promotional effort per dollar is different for different competitors.
4. Each competitor's share of market depends on his relative effective promotional effort and the difference between his price and average price of all other competitors.
5. Collusion is not permitted and each competitor tries to maximize his net revenue or payoff.

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31. McCarthy, Basic Marketing: A Managerial Approach, Homewood, III: R. D. Irwin, 3rd ed., 1968.

Total market potential represents total sales of all competitors under a given set of strategies. In general the market potential is a function of price and promotional efforts of all competitors and such exogeneous variables as population, national income, etc.

The contribution of promotional effort to the market share of xj is proportional to $(\alpha_j S_j)^{e_j} / \sum_{i=1}^{i=n} (\alpha_i S_i)^{e_i}$ where Si is the expenditure on promotional effort, α_i is the effectiveness of promotional effort per dollar of Xi, and ei is a positive constant $i = 1, 2, \ldots, n$.

The contribution of price to the share of market of Xj is assumed to be kj $(\bar{p}_j - p_j)$ where pj is the price charged by Xj and $\bar{p}_j$ is the average price of all other competitors. 32

Though kj maybe different for different competitors, in our assumption, it is assumed that all kj's are the same, and the common value is denoted by k. Thus the contribution of price to Xj's market share is $k(\bar{p}_j - p_j)$.

The quality of product, distribution channels, and other factors that influence sales are assumed to be fixed for each competitor. The contribution of all these factors is denoted by a constant vj for Xj. Thus, the share of market $\int_j$ of Xj is given by

$$\int_j = v_j + u(\alpha_j S_j)^{e_j} / \sum_{i=1}^{i=n} (\alpha_i S_i)^{e_i} + k/(n-1) \left(\sum_{i=1}^{i=n} p_i - np_j \right),$$

where u is a positive constant such that

$$\sum_{i=1}^{i=n} \int_i = \sum_{i=1}^{i=n} v_i + u = 1$$

conflicts with next page!

Thus, $\int_i$'s must be between 0 and 1 for all values of pi and Si.

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32. If $\bar{p}$ is average price of all the n-competitors then the contribution of price to the share of market can also be written as

$$k_j^1(\bar{p} - p_j) = k_j^1 \left[(1/n) \sum_{i=1}^{i=n} p_i - p_j \right]$$
$$= k_j^1 \left\{ (1/n) \sum_{i \neq j} p_i - [(n-1)/n] p_j \right\}$$
$$= k_j^1 [(n-1)/n] \left[1/(n-1) \sum_{i \neq j} p_i - p_j \right]$$
$$= k_j(\bar{p}_j - p_j) \text{ where } k_j = k_j^1 (n-1)/n.$$

Each competitor has infinite number of strategies. It is assumed that each competitor is fully aware of the objective functions of all other competitors and their strategies. Also, gain for one competitor does not necessarily mean loss for any other competitor. It is possible for all competitors to increase (or decrease) their profits at the same time. Thus the problem of maximizing the profits of n-competitors is an n-person non-zero sum game. Thus, in some particular cases, equilibrium solutions are obtained under the assumptions stated earlier.

FORMULATION OF THE PROBLEM

The payoff (net revenue) for Xj is given by

Payoff = (total volume of sales) x (Profit margin) - (promotional expenditure)

$$Rj = \begin{cases} Af jmj - Sj & \text{if } \sum_{i=1}^{i=n} Si > 0, \text{*33} \\ 0 & \text{if } S_1 = S_2 = S_3 \ldots = Sn = 0, \end{cases}$$

where, for Xj,

Rj = net revenue

A = market potential which is a function of price and promotional efforts of all competitors,

$f j = vj + uxj + k/(n-1) \left(\sum_{i=1}^{i=n} pi - npj \right)$,

pj = price,

sj = expenditure on promotional effort

$xj = (\alpha j\ Sj)^{ej} / \sum_{i=1}^{i=n} (\alpha i\ Si)^{ei}$,

$mj = pj - cj$ = margin of profit

cj = unit cost of production,

$ei, \alpha i, Vi, u,$ and k are all constants and

$$\sum_{i=1}^{i=n} Vi + u = 1$$

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*33. $\sum_{i=1}^{i=n} Si > 0$ implies that at least one competitor spends on promotional effort.

It is assumed that production is equal to sales of any competitor. The problem is to find equilibrium points (p^*, s^*) for each competitor in the sense that if any competitor deviates from the equilibrium values, his payoff goes down. The equilibrium values of pj and sj are obtained by maximizing payoff for Xj subject to the constraints that $Sj \geq 0$ and $mj \geq 0$.

Assuming the above relation between payoff and variables affecting it, the following four models are discussed.

(a) Model I: A is independent of price and promotional effort.

(b) Model II: A depends on promotional effort only.

(c) Model III: A depends on price only.

(d) Model IV: A depends on both price and promotional effort.

SOLUTIONS OF THE MODELS

(a) Model I: Market Potential Is Independent of Price and Promotional Effort.

Applying Kuhn-Tucker conditions, the maximum value of Rj subject to $Sj \geq 0$ is obtained from the following equations.[34]

$$\partial Rj/\partial pj = 0,$$
$$\partial Rj/\partial sj \leq 0, \qquad (1)$$
$$sj\,\partial Rj/\partial sj = 0.$$

If $Sj > 0$, then $\partial Rj/\partial sj = 0$. Equation (1) is solved for the following cases.

(i) $Sj > 0$;
(ii) $Sj = 0$; $\sum_{i=1}^{i=n} Si > 0$

$\partial Rj/\partial pj = 0$ gives (2)

$$\int j = kmj = vj + uxj + [k/(n-1)] \left(\sum_{i=1}^{i=n} pi - npj\right)$$

$\partial Rj/\partial sj = 0$ gives $Sj = Amjuejxj\,(1 - xj)$ (3)

Solving equations (2) and (3) for j = 1, 2, ..., n the solutions corresponding to $Sj > 0$ and $mj > 0$ are given by (asterisks denote these values):

$$\sum_{i=1}^{i=n} pi^* = \sum_{i=1}^{i=n} Ci + 1/k,$$

$$mj^* + [(n-1)(vj + uxj^*) + 1 + \lambda j]/ k(2n-1)$$

where $\lambda j = k\left(\sum_{i=1}^{i=n} Ci - ncj\right)$,

$$Sj^* = Auejmj^*xj^*(1 - xj^*),$$

$$Rj^* = Akm^2j^* - sj^*,$$

where $Sj = 0$ and $\sum_{i=1}^{i=n} Si > 0$, then

$$Rj = A[vj + k/(n-1)\left(\sum_{i=1}^{i=n} pi - npj\right)]\,mj,$$

$$Ri = A\left[(vi + uxi + k/(n-1)\left(\sum_{W=1}^{W=n} pw - npi\right)\right]mi - Si$$

when $(i \neq j)$

By differentiating Rj and Ri with respect to the respective controllable variables and simplifying the solutions corresponding to Sj = 0 ane given by (** denote these solutions):

$$mj^{**} = \left[(n-1)\,vj + 1 + \lambda j\right] / k(2n-1),$$

$$mi^{**} = \left[(n-1)(vi + uxi^{**}) + 1 + \lambda i\right] / k(2n-1),$$

$$Si^{**} = Aueimi^{**}\,xi^{**}\,(1 - xi^{**}),$$

$$Rj^{**} = Akm^2j^{**} \cancel{- Si^{**}}.$$

$$Ri^{**} = Akm^2j^{**} - Si^{**}$$

If $S_1 = S_2 = \ldots\ldots = S_n = 0$, then the payoff for each competitor is zero.

Particular Case when $\alpha_1 = \alpha_2 \ldots = \alpha n$ and $e_1 = e_2 = \ldots\ldots en = 1$

The payoff for Xj is

$$Rj = A\left[vj + usj/\sum_{i=1}^{i=n} Si + k/(n-1)\right.$$

$$\left.\left(\sum_{i=1}^{i=n} pi - npj\right)\right] mj - Sj.$$

If $Sj > 0$ then Rj is maximum when

$$mj^{*} = \left[(n-1)(vj + uxj) + 1 + \lambda j\right] / k(2n-1) \qquad (4)$$

$$sj^{*} = Aumj^{*}\,xj^{*}\,(1 - xj^{*}).$$

If $Sj = 0$ and $\sum_{i=1}^{i=n} Si > 0$, then Rj is maximum when

$$mj^{**} = \left[(n-1)\,vj + 1 + \lambda j\right] / k(2n-1). \qquad (5)$$

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34. The set of equations given by (1) is also sufficient for maximum Rj if Rj is a concave function. Now Rj is concave if

$$\begin{bmatrix} \partial^2 Rj/\partial pj^2, & \partial^2 Rj/\partial pj\,\partial Sj, \\ \partial^2 Rj/\partial pj\,\partial Sj, & \partial^2 Rj/\partial^2 Sj^2, \end{bmatrix}$$ is negative definite

The equilibrium values are obtained from (4) and (5) by the procedure given below.

The procedure consists of finding equilibrium values for two competitors [35] and extending it to n-competitors successively. In the following procedure, it is assumed that $t = 2$, these exist S_1* and S_2* that satisfy equation (4).

In the case of two competitors, at each stage it has been tested whether the incoming competitor can profit more by spending on promotional effort. We can do so similarly in this case.

(1) Relabel the competitor so that

$$(n-1)v_1 + \lambda_1 \geq (n-1)v_2 + \lambda_2 \geq \ldots \geq (n-1)\,v_n + \lambda_n$$

(2) Find an initial $(S_1*, S_2*, \ldots, S_t*, 0, 0, \ldots, 0)$ where t represents the number of competitors who spend on promotional effort and S_i*'s satisfy equation (4) Initially $t = 2$.

(3) If $t < n$, test whether $Au/k(2n-1)$

$$[(n-1)\,v_{t+1} + \lambda_{t+1}] \geq \sum_{i=1}^{i=t} Si*;$$

(4) If yes, go to step (2) and replace t by $(t + 1)$.

(5) If no, $(S_1*, S_2*, \ldots, S_t*, 0, \ldots 0)$ is the equilibrium solution.

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35. For detailed discussions on the existence of equilibrium solutions for the case of two competitors, see, Shiv. Gupta, "Mathematical Game Model for a Duopolistic Market", Management Science, 13, 568-583 (1967).

(b) Model II: Total Market Potential Is a Function of Promotional Effort Only.

In this model, the total market potential is a function of total effective promotional efforts of all competitors and is independent of prices. If $Z = \sum_{i=1}^{i=n} \alpha_i S_i$ where $\alpha_i Si$ is the effective promotional effort of Xi, then the total market potential, A, is a function of 2 and is independent of prices.

Hence, $\partial A / \partial pj = 0$

$$\partial A / \partial Sj = \alpha j \; dA / dz$$

If Rj is a concave function of pj and Sj, then the Kuhn-Tucker condition will give maximum value of Rj for a given set of values of pi and Si $(i \neq j)$.

The concavity of Rj will depend upon the functional form of A.* If $Sj > 0$ and $mj > 0$, the maximum value of Rj is given by

$$\partial Rj / \partial pj = 0,$$

$$\partial Rj / \partial Sj = 0.$$

If $Sj = 0$ $\sum_{i=1}^{i=n} Si > 0$ and $mj > 0$, then the maximum Rj is given by

$$\partial Rj / \partial pj = 0$$

$$Sj = 0$$

If $S_1 = S_2 = \ldots = S_n = 0$, the net revenue $Rj = 0$.

Let $A = a \left[1 - \exp(-\gamma z) \right]$ where a and Y are positive constants, the solutions are as follows:

If $Sj > 0$, $mj > 0$ the maximum value of Rj is obtained for:

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* If Rj is concave, Kuhn-Tucker conditions lead to maximum value of Rj. In all subsequent discussions, it is assumed that Kuhn-Tucker conditions lead to maximum value of Rj.

$$mj^* = [(n-1)(vj + uxj) + 1 + \lambda j] / k(2n-1),$$

$$Sj^* = Auejmj^*xj^* (1 - xj^*) + kmj^{2*} \alpha jy (a - A)$$

if $Sj - 0$, $\sum_{i=1}^{i=n} Si > 0$ and $mj > 0$, the maximum value of Rj is obtained for

$$mj^{**} = [(n-1) vj + 1 + \lambda j] / k(2n-1).$$

By comparing profits in the two situations, the equilibrium solutions can be found.

(c) Model III: Total Market Potential Is a Function of Price Only.

In this model the total market potential is a function of price only, i.e. $\partial A / \partial Sj = 0$ for $j = 1, 2, \ldots, n$.

If $Sj > 0$ and $mj > 0$, the maximum value of Rj is obtained when

$$\partial Rj / \partial pj = 0$$

and $$\partial Rj / \partial Sj = 0 \qquad (6)$$

If $Sj = 0$, $\sum_{i=1}^{i=n} Si > 0$ and $mj > 0$, then maximum value of Rj is given by

$$\partial Rj / \partial pj = 0 \qquad (7)$$

When $A = a [1 - \sum_{i=1}^{i=n} wipi]$,

equations (6) and (7) simplify as follows:

If $Sj > 0$ and $mj > 0$,

$\partial Rj / \partial pj = 0$ gives

$$mj^* = (A - awjmj^*) [vj + uxj^* + k/(n-1) \times (\sum_{i=1}^{i=n} pi^* - nj^*)] / Ak \qquad (8)$$

$\partial Rj \partial Sj = 0$ gives

$$Sj^* = Auejmj^*xj^* (1 - xj^*).$$

For $j = 1, 2, \ldots, n$ equations (8) and (9) represent 2n equations involving 2n unknowns, and hence the values of pj and Sj for $j = 1, 2, \ldots, n$ can be obtained.

If $Sj = 0 \sum_{i=1}^{i=n} Si > 0$ and $mj > 0$ then

$$mj^{**} = (A - awjmj^{**}) [vj + k/(n - 1)$$

$$x \left(\sum_{i=1}^{i=n} Pi^{**} - npj^{**} \right)] / Ak \tag{10}$$

$$(j = 1, 2, \ldots, n)$$

Equations (10) gives n equations involving n unknowns and hence they can be solved simultaneously. The equilibrium solutions can be obtained by comparing the profits in the two situations.

(d) Model IV: Total Market Potential Is a Function Of Both Price And Promotional Effort.

In this model the total market potential is a function of both price and promotional efforts of all competitors. The equilibrium solutions of Xj are obtained, as before, by comparing maximum payoff when $Sj > 0$ with that when $Sj = 0$

Let $A = a [1 \sum_{i=1}^{i=n} wipi] [1 - \exp(-\gamma z)]$,

where

pi = price charged by Xi,

$Z = \sum_{i=1}^{i=n} (\alpha i\ Si)$,

and a, wi, γ are constants. If $Sj > 0$ and $mj > 0$, the maximum value of Rj is obtained when:

$$(1 - wjmj^* - \sum_{i=1}^{i=n} wipi^*) [vj + uxj^* + k/(n - 1)$$

$$x \left(\sum_{i=1}^{i=n} pi^* - npj^* \right)]$$

$$= kmj^* (1 - \sum_{i=1}^{i=n} wipi^*),$$

$$Sj^* = a\left(1 - \sum_{i=1}^{i=n} wipi^*\right) mj^* \left[uejxj^*(1 - xj^*) + \gamma\alpha jSj^* \exp(-\gamma z^*) f j^* \right]. \quad (j = 1, 2, \ldots, n) \qquad (11)$$

If $Sj = 0$, $\sum_{i=1}^{i=n} Si > 0$ and $mj > 0$, then the maximum value of Rj is obtained when

$$(A - awjmj^{**}) f j^{**} = Akmj^{**}, \quad (j = 1, 2, \ldots, n) \qquad (12)$$

Equation (11) gives a set of 2n equation involving 2n unknowns and equation (12) gives a set of n equations involving n unknowns. Hence these equations can be solved for the unknowns. By comparing the profits, the equilibrium solution can be found.

Numerical Example:

Example: The following numerical example gives the equilibrium solutions for the four models considered.

$n = 2,\ c_1 = 50,\ c_2 = 51,\ \alpha_1 = \alpha_2 = 1,\ e_1 = e_2 = 1,$

$v_1 = v_2 = 0,\ u = 1,\ k = 0.1$

Model I: $A = 1{,}000,$

Model II: $A = a[1 - \exp\{-\gamma(\alpha_1 s_1 + \alpha_2 s_2)\}],$

$a = 1{,}000 \qquad \gamma = 0.001$

Model III: $A = a[1 - w_1 p_1 - w_2 p_2],$

$a = 1{,}000,\quad w_1 = w_2 = 0.005$

Model IV: $A = a[1 - \exp\{-\gamma(\alpha_1 s_1 + \alpha_2 s_2)\}]$

$\times (1 - w_1 p_1 - w_2 p_2),$

$a = 1{,}000,\quad \gamma = 0.001,\quad w_1 = w_2 = 0.005.$

The equilibrium values are obtained from the following equations when either or both of s_1 and s_2 are zero, the maximum values of R_1 and R_2 are obtained from the same set of equations given below for all the four models.

When $s_1 = 0$ and $s_2 > 0$,

$$m_1^{**} = (1.1) / 0.3,$$

$$m_2^{**} = (1.9) / 0.3,$$

$s_1^{**} = 0$, s_2^{**} = some positive quantity.

When $s_1 > 0$ and $s_2 = 0$,

$$m_1^{**} = (2.1) / 0.3,$$

$$m_2^{**} = (0.9) / 0.3,$$

s_1^{**} = some positive quantity

$$s_2^{**} = 0.$$

When $s_1 = 0$ and $s_2 = 0$;

m_1^{**} and m_2^{**} are arbitrary.

When $s_1^{**} > 0$ and $s_2^{*} > 0$ the maximum values of R_1 and R_2 are obtained from the following equation:

<u>Model I</u>:

$$m_1^{*} = (1.1 + x_1^{*}) / 0.3,$$

$$m_2^{*} = (0.9 + x_2^{*}) / 0.3,$$

$$m_1^{*} (1 - x_1^{*}) = m_2^{*} (1 - x_2^{*}) = (s_1^{*} + s_2^{*}) / 1000.$$

<u>Model II</u>:

$$m_1^{*} = (1.1 + x_1^{*}) / 0.3,$$

$$m_2^{*} = (0.9 + x_2^{*}) / 0.3,$$

$$0.1s^{*}m_1^{*2}\exp(-0.001s^{*})$$

$$+ 1000 [1 - \exp(-0.001s^{*})] m_1^{*} (1 - x_1^{*}) = s^{*},$$

$$0.1s^{*}m_2^{*2}\exp(-0.001s^{*})$$

$$+ 1000 [1 - \exp(-0.001s^{*})] m_2^{*} (1 - x_2^{*}) = s^{*}$$

$$\text{Model III:}\quad (1000 - 5(p_1^* + p_2^*) - 5m_1^*]\,[10x_1^* + p_2^* - p_1^*]$$
$$= [1000 - 5(p_1^* + p_2^*)]m_1^*,$$
$$[1000 - 5(p_1^* + p_2^*) - 5m_2^*]\,[10x_2^* + p_1^* - p_2^*]$$
$$= [1000 - 5(p^* + p_2^*)]\, m_2^*,$$
$$m_1^*\,(1 - x_1^*) = m_2^*\,(1 - x_2^*) = s_1^* + s_2^*)\,/\,1000.$$

$$\text{Model IV:}\quad 3p_1^{*2} - 2p_1^*\,(250 + 10x_1^*) + 500(20 + x_1^*)$$
$$+ (200 - p_2^*)\,(10x_1^* + p_2^*) = 0,$$
$$3p_2^{*2} - 2p_2^*(251 + 10x_2^*) + 510\,(20 + x_2^*)$$
$$+ (200 - p_1^*)\,(10x_2^* + p_1^*) = 0,$$
$$\exp\,[- 0.001\,(s_1^* + s_2^*)]\,[(10x_1^* + p_2^* - p_1^*)\,/$$
$$10{,}000 - x_2^*\,/\,(s_1^* + s_2^*)\,]$$
$$+ x_2^*\,/\,(s_1^* + s_2^*) = 1/5(p_1^* - 50)\,(200 - p_1^*\,p_2^*),$$
$$\exp\,[- 0.001\,(s_1^* + s_2^*)]\,[10x_2^* + p_1^* - p_2^*)\,/$$
$$10{,}000 - x_1^*\,/(s_1^* + s_2^*)]$$
$$+ x_1^*\,/\,(s_1^* + s_2^*) = 1/5(p_2^* - 51)\,(200 - p_1^* - p_2^*).$$

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